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THE EQUILIBRIUM OF SHORT STRAIN-HARDENING STEEL COLUMNS

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Abstract—The post buckling behavior of short, centrally loaded, structural steel columns subjected to monotonically increasing axial deformation is predicted analytically using a simplified Shanley model and the numerical results obtained are compared with experimental data. Because of the presence of the plastic flow and strain-hardening regions in the stress–strain relation, this behavior is very complex. It is shown that columns having slenderness ratios less than a certain critical value can, after buckling laterally a limited amount, actually return to their straight position prior to buckling to complete failure.

NOTATION

The following symbols are used in this paper:

A = sectional area of the cell.
a = half length of the cell.
b = half width of the cell, radius of gyration, core radius.
E = Young's modulus.
e_c, e_t = change of the length of cell elements.
E_{st} = strain hardening modulus.
L = length of the column.
M_e = external bending moment.
ΔM_e = increment of the external bending moment.
M_i = internal bending moment.
ΔM_i = increment of the internal bending moment.
P = axial thrust.
ΔP = increment of the axial thrust.
ΔP_c = increment of the force in element c.
$P_E = 2AE/\alpha\lambda^2$ = Euler load.
$P_R = 4AE_{st}/\alpha\lambda^2$ = reduced modulus load.
$P_T = 2AE_{st}/\alpha\lambda^2$ = second tangent modulus load.
ΔP_t = increment of the axial force in element t.
$P_u = A\sigma_u$ = compressive maximum strength of the stub-column.
P_y = axial yield load.
$\alpha = a/L$.
$\gamma = P_y/P_R$.
δ = lateral deflection of the column.
$\delta' = \Delta\delta/b$ = nondimensional deflection of the column.
$\Delta\delta$ = increment of the column deflection.
$\Delta\delta' = \Delta\delta/b$ = nondimensional deflection increment.
ϵ_c = strain in element c.
$\Delta\epsilon_c$ = inelastic strain increment in element c.
$\epsilon_0 = \epsilon_{st} - \epsilon_y$.
ϵ_{st} = strain at strain hardening point.
ϵ_t = strain in element t.
$\Delta\epsilon_t$ = inelastic strain increment in element t.
ϵ_y = strain at yield point.
$\lambda = L/b$ = slenderness ratio of the column.
σ_u = tensile strength or compressive maximum strength of the material.
σ_y = yield point.

Subscripts 1, 2, 3—attached to P, ΔP and $\delta(\delta')$, $\Delta\delta(\Delta\delta')$ indicate that those values are of terminal point of Regions 1, 2, 3—respectively.

INTRODUCTION

The post buckling behavior of short centrally loaded columns depends greatly on the stress–strain characteristics of the material. In the case of structural steel, the elastic, plastic-flow, and strain hardening regions define the main characteristics involved. When a column of this material is subjected to monotonically increasing axial deformation, its flexural rigidity reduces to zero as the axial stress builds up to the yield value. When this point is reached, the column will immediately buckle laterally indicating the bifurcation point has been reached. The column will maintain static equilibrium with increasing axial deformation but the axial load will gradually decrease from its initial yield value. After a certain amount of axial deformation, strain-hardening in the material will occur causing the axial load to increase with further increases in axial deformation. If the column is very short, the earlier loss in load carrying capacity will be completely restored as the axial deformation is increased and the ultimate load carrying capacity eventually reached will be considerably larger than the initial yield load.

During post buckling of very short columns, the lateral displacement pattern is quite complex. For example, after a limited amount of buckling, the column can actually return to its original straight position prior to buckling a second time to complete failure.

To provide a better understanding of the post buckling behavior described above, analytical relations for axial load vs lateral deflection up to complete failure are derived herein using a simplified mathematical model similar to that of Shanley [1]. The characteristic behavior obtained from this model is then compared with the results of tests conducted on H-shaped steel columns.

It is believed that the findings of this study will shed considerable light on other types of inelastic buckling, e.g. the coupled flexural–torsional buckling of steel beams and beam columns which is such an important consideration in plastic design or in aseismic design. Based on experimental observation, Galambos [2, 3] first pointed out that the initiation of flexural–torsional buckling of an H-shaped member does not disturb the development of its in-plane plastic moment provided the slenderness of the member is sufficiently small. Lay [4, 5] then treated this problem theoretically and introduced the intuitive concept of "dynamic jump of strain in plastic region". Since lateral buckling can be thought of as the flexural buckling of a flange in its own plane, the present analysis explains the physical meaning of Lay's intuitive assumption.

Another type of inelastic buckling which can be better understood with the present analysis is the local flange buckling of H-shaped sections. In this case, a phenomenon similar to the post buckling behavior described above is observed, namely, a flange will start to wrinkle as soon as the compressive stress reaches the yield point but the amplitude of the wrinkle will not continue to grow with increases in loading of the member; thus, the overall load carring capacity of the member will not be reduced provided the width-to-thickness ratio of the flange is within certain limits [6, 7]. Although this is a problem of plate instability, the fundamental mechanism of equilibrium is similar to that shown by the present analysis.

ANALYTICAL MODEL AND INITIAL CRITICAL LOAD

As previously pointed out, the post buckling behavior of short centrally loaded columns depends greatly on the stress–strain characteristics of the material. Structural steel is characterized by the elastic, plastic flow, and strain-hardening regions of the stress–strain relationship. In the present study, this relationship is simplified by assuming the rigid–plastic flow–strain hardening relation shown in Fig. 1 by a bold line where σ_y is the yield stress, ϵ_y is the corresponding yield strain, and ϵ_{st} is the strain at which strain hardening begins. The strain hardening modulus E_{st} is assumed to remain constant in this relation.

The centrally loaded steel column having both ends hinged is modelled mathematically using

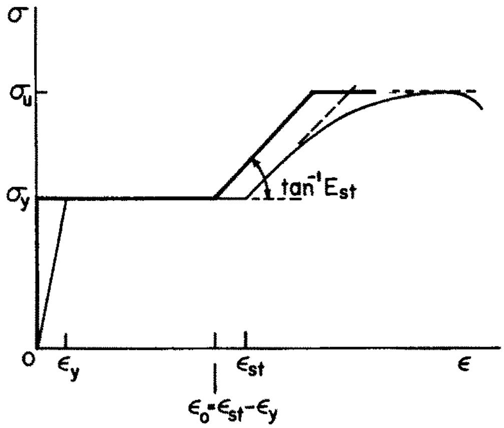


Fig. 1. Stress–strain relation.

the Shanley model shown in Fig. 2. This model has two rigid arms inter-connected by a cell consisting of two flange elements each having a cross sectional area $A/2$. It is assumed that buckling takes place in the web plane only and that shear deformations can be ignored.

The initial critical load of the column is easily determined from the model by equating the external and internal bending moments. As indicated in Fig. 2(b), e_c and e_t denote the changes in length of cell elements c and t, respectively, which occur after the start of bending. The resulting geometry changes cause a lateral deflection as given by

$$\delta = \frac{L}{2}\theta = \frac{L}{2}\left(\frac{e_c + e_t}{4b}\right) = \frac{aL}{4b}(\epsilon_c + \epsilon_t)$$

where $\epsilon_c = e_c/2a$ and $\epsilon_t = e_t/2a$ are the strains in flange elements c and t, $2a$ equals the cell length, and b is the half-width of the cell equal to the radius of gyration and also equal to the core

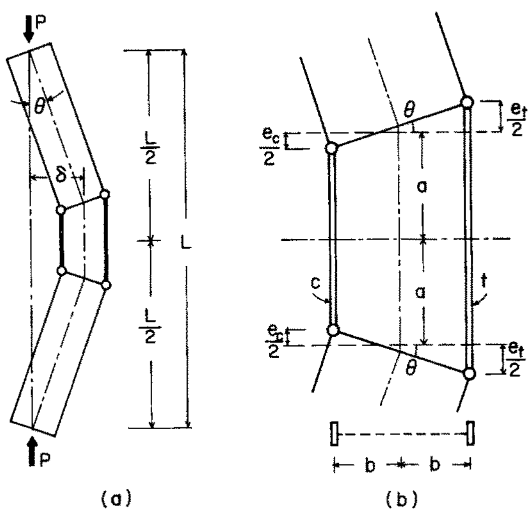


Fig. 2. Shanley's model.

radius in the model. The external bending moment at the hinge is

$$M_e = P\delta = \frac{aLP}{4b}(\epsilon_c + \epsilon_t)$$

while the internal bending moment about the hinge point is

$$M_i = \frac{Ab}{2}(E_c\epsilon_c + E_t\epsilon_t)$$

where E_c and E_t are the effective moduli of their respective elements. Equating internal and external bending moments, one obtains

$$P = \frac{2b^2A}{aL}\left(\frac{E_c\epsilon_c + E_t\epsilon_t}{\epsilon_c + \epsilon_t}\right) = \frac{2A}{\alpha\lambda^2}\left(\frac{E_c\epsilon_c + E_t\epsilon_t}{\epsilon_c + \epsilon_t}\right) \tag{1}$$

where $\lambda = L/r = L/b$ is the slenderness ratio and $\alpha = a/L$. When buckling occurs in the elastic range, $E_c = E_t = E$ in which case Eq. (1) reduces to the Euler form

$$P_E = \frac{2AE}{\alpha\lambda^2}. \tag{2}$$

If P_E is larger than the axial yield load for the column, elastic buckling cannot occur; however, inelastic buckling can occur starting as soon as the stresses in the flange elements of the cell reach the yield value. Buckling is initiated at this instant of loading since no bending stiffness is present in the member ($E_c = E_t = 0$). Thus, the initial critical load of a short column is always equal to the axial yield load for the section, namely, $P_y = A\sigma_y$. Of principal interest here however is the post buckling behavior of such members. Therefore, the subsequent analytical treatment considers the post yield behavior of columns having slenderness ratios $\lambda \leq \sqrt{2E/\alpha\sigma_y}$.

POST BUCKLING BEHAVIOR

A. *Region 1—Strain in element* c *increases from ϵ_y to ϵ_{st} while elastic strain reversal takes place in element* t

Immediately after the column starts to deflect laterally under the action of an axial yield thrust, strain reversal must take place in element t to maintain static equilibrium. Clearly, if the strains in both elements were to increase into the plastic flow range, there could be no change in the internal bending moment to balance the change in external bending moment caused by deflection of the column. Thus, the strain reversal in element t must take place.

Consider the increment of hinge rotation $\Delta\theta$ from the critical straight state as shown in Fig. 3. The increment of strain in element t, which is the elastic strain reversal, is shown equal to zero in this figure consistent with the infinite modulus of the material as represented by the rigid portion of the rigid–plastic flow–strain hardening relationship. The increment of lateral deflection of the column is given by

$$\Delta\delta = \frac{L}{2}\Delta\theta = \frac{aL}{4b}\Delta\epsilon_c = \frac{\alpha\lambda^2}{4}b\,\Delta\epsilon_c$$

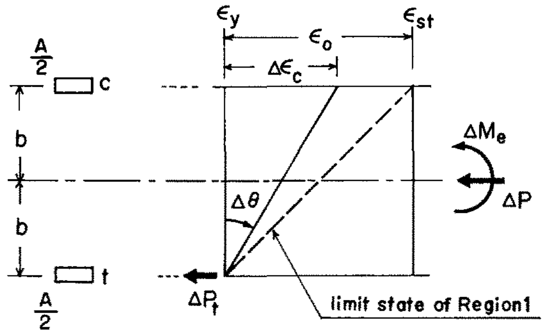


Fig. 3. Region 1.

where $\Delta\epsilon_c$ is the inelastic strain increment in element c. The increment of internal bending moment about the hinge point is

$$\Delta M_i = -\Delta P_t b = -\Delta P b$$

where ΔP_t is the increment of axial force in element t and ΔP is the increment of external axial thrust. These increments of force are taken as positive when they represent increasing compressive forces. The increment of external bending moment at the hinge is

$$\Delta M_e = (P_y + \Delta P)\Delta\delta.$$

Equating the increments of internal and external bending moments, one obtains,

$$\Delta P = \frac{-P_y \Delta\delta}{b + \Delta\delta};$$

thus, the axial thrust P for a given deflection $\Delta\delta$ becomes

$$P = P_y + \Delta P = \left(\frac{b}{b + \Delta\delta}\right)P_y. \tag{3}$$

Introducing the following nondimensional expression

$$\Delta\delta' = \frac{\Delta\delta}{b} = \frac{\alpha\lambda^2}{4}\Delta\epsilon_c,$$

Eq. (3) can be written as

$$P = \left(\frac{1}{1 + \Delta\delta'}\right)P_y. \tag{3$'$}$$

Equations (3) and (3)′ are valid until $\Delta\epsilon_c$ reaches the value $\epsilon_0 = \epsilon_{st} - \epsilon_y$. At this terminal point, the lateral deflection δ_1(or δ_1'), the increment of axial thrust ΔP_1, and the total axial thrust P_1 are given by the relations

$$\delta_1 = (\Delta\delta)_{\Delta\epsilon_c=\epsilon_0} = \frac{\alpha\lambda^2}{4} b\epsilon_0$$

or

$$\delta_1' = \frac{\alpha\lambda^2}{4}\epsilon_0 \tag{4}$$

$$\Delta P_1 = \Delta P_t = \left(\frac{-\delta_1'}{1+\delta_1'}\right)P_y \tag{5}$$

$$P_1 = \left(\frac{1}{1+\delta_1'}\right)P_y \tag{6}$$

where

$$\delta_1' = \delta_1/b.$$

B. *Region 2—Strain in element* c *increases into the strain hardening region while the strain in element* t *remains in the elastic region*

The increments of strain, deflection, and forces are measured from the terminal state of Region 1 as shown in Fig. 4. In this case, one obtains the relations

$$\Delta\delta = \frac{\alpha\lambda^2}{4} b\Delta\epsilon_c, \quad \text{or} \quad \Delta\delta' = \frac{\alpha\lambda^2}{4}\Delta\epsilon_c \tag{2-1}$$

$$\Delta P_c = \frac{AE_{st}}{2}\Delta\epsilon_c \tag{2-2}$$

$$\Delta P = \Delta P_c + \Delta P_t \tag{2-3}$$

$$\Delta M_i = (\Delta P_c - \Delta P_t)b = (AE_{st}\Delta\epsilon_c - \Delta P)b \tag{2-4}$$

$$\begin{aligned}\Delta M_e &= (P_1 + \Delta P)(\delta_1 + \Delta\delta) - P_1\delta_1 \\ &= P_1\Delta\delta + \Delta P\delta_1 + \Delta P\Delta\delta\end{aligned} \tag{2-5}$$

where ΔP_c denotes the increment of force in element c. Equating the above increments of

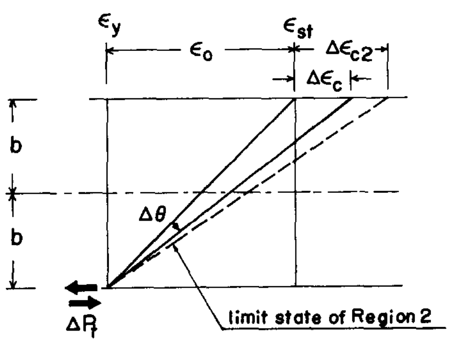


Fig. 4. Region 2.

internal and external bending moments gives

$$AE_{st}\Delta\epsilon_c - \Delta P = P_1\Delta\delta' + \Delta P\delta_1' + \Delta P\Delta\delta'$$

from which

$$\Delta P = \frac{(P_R - P_1)\Delta\delta'}{1 + \delta_1' + \Delta\delta'}$$

where

$$P_R = \frac{4AE_{st}}{\alpha\lambda^2}$$

represents a reduced modulus load. Substituting for P_1 from Eq. (6), one obtains

$$\Delta P = \left[\frac{(1 + \delta_1' - \gamma)\Delta\delta'}{\gamma(1 + \delta_1')(1 + \delta_1' + \Delta\delta')}\right]P_y \tag{7}$$

where

$$\gamma = P_y/P_R = \frac{\alpha\lambda^2}{4}\left(\frac{\sigma_y}{E_{st}}\right).$$

The axial thrust can now be expressed as

$$P = P_1 + \Delta P = \left[\frac{\gamma + \Delta\delta'}{\gamma(1 + \delta_1' + \Delta\delta')}\right]P_y. \tag{8}$$

Using Eq. (2-3), the force increment in element t becomes

$$\Delta P_t = \Delta P - \frac{AE_{st}}{2}\Delta\epsilon_c = \Delta P - \frac{1}{2}P_R\Delta\delta'.$$

Substituting for ΔP from Eq. (7) results in the relation

$$\Delta P_t = \frac{P_y}{2\gamma}\left[\frac{(1 - 2\gamma - \delta_1'^2) - (1 + \delta_1')\Delta\delta'}{(1 + \delta_1')(1 + \delta_1' + \Delta\delta')}\right]\Delta\delta'. \tag{9}$$

From the form of Eq. (9), three sub regions can be defined as follows:

(1) *Sub-region* 2-*A*. Note that when the relation

$$1 - 2\gamma - \delta_1'^2 \leq 0 \tag{10}$$

is satisfied, ΔP_t as given by Eq. (9) is negative for positive values of $\Delta\delta'$. This condition means that the strain reversal in element t continues with increasing axial deformation to the point of complete failure of the column. The entire post-yield behavior of the column is therefore expressed by Eq. (8). Note that the terminal strength of the column as $\Delta\delta'$ approaches infinity is P_R.

(2) *Sub-region* 2-*B*. If the term $(1 - 2\gamma - \delta_1'^2)$ is positive, ΔP_t as given by Eq. (9) will also be

positive when $\Delta\delta'$ falls in the range

$$0 < \Delta\delta' < \frac{(1 - 2\gamma - \delta_1'^2)}{(1 + \delta_1')}.$$

The peak value of ΔP_t within this range corresponds to that value given by Eq. (9) when $\Delta\delta'$ satisfies the maximum condition

$$\frac{d(\Delta P_t)}{d(\Delta\delta')} = 0;$$

thus, one obtains

$$(\Delta P_t)_m = \frac{P_R}{2}\left\{\frac{[\sqrt{2(1+\delta_1'-\gamma)} - (1+\delta_1')]^2}{1+\delta_1'}\right\} \tag{11}$$

$$(\Delta\delta')_m = \sqrt{2(1+\delta_1'-\gamma)} - (1+\delta_1'). \tag{12}$$

If $(\Delta P_t)_m$ given by Eq. (11) is smaller than the decrease of compressive load undergone at the terminal state of Region 1,

$$\frac{P_R}{2}\left\{\frac{[\sqrt{2(1+\delta_1'-\gamma)} - (1+\delta_1')]^2}{1+\delta_1'}\right\} < \frac{\delta_1' P_y}{1+\delta_1'}$$

which reduces to

$$(1 - 2\gamma - \delta_1')^2 - 8\gamma\delta_1' < 0, \tag{13}$$

then the stress in element t never reaches the yield point and the post-yield behavior of the column is again defined by Eq. (8).

(3) *Sub-region* 2-*C*. If on the other hand, the condition

$$(1 - 2\gamma - \delta_1')^2 - 8\gamma\delta_1' \geq 0 \tag{14}$$

is satisfied, the stress in element t can recover the decrease given by Eq. (5); thus, it will return once again to the yield level. Since Eqs. (7)–(9) are valid only when the strain in element t remains in the elastic region, the terminal equilibrium state of Region 2 is obtained by equating the value of ΔP_1 given by Eq. (9) to the negative value of ΔP_1 given by Eq. (5). Therefore, one obtains the terminal condition

$$\frac{P_Y}{2\gamma}\left[\frac{(1 - 2\gamma - \delta_1'^2) - (1+\delta_1')\Delta\delta'}{(1+\delta_1')(1+\delta_1'+\Delta\delta')}\right]\Delta\delta' = \left(\frac{\delta_1'}{1+\delta_1'}\right)P_y$$

which reduces to

$$\Delta\delta_2'^2 - (1 - 2\gamma - \delta_1')\Delta\delta_2' + 2\gamma\delta_1' = 0 \tag{15}$$

where $\Delta\delta_2'$ is the nondimensional deflection increment at the terminal state of Region 2. The corresponding strain increment in element c is

$$\Delta\epsilon_{c\cdot 2} = \left(\frac{4}{\alpha\lambda^2}\right)\Delta\delta_2'. \tag{16}$$

Replacing $\Delta\delta'$ in Eq. (8) by $\Delta\delta_2'$ and making use of Eq. (15), the axial thrust at the terminal state becomes

$$P_2 = \left(\frac{\Delta\delta_2'}{2\gamma\delta_2'}\right)P_y \tag{17}$$

where δ_2' is the total nondimensional deflection at the terminal state of Region 2, namely

$$\delta_2' = \delta_1' + \Delta\delta_2'. \tag{18}$$

C. *Region 3—Plastic flow takes place in element* t *while elastic strain reversal takes place in element* c

If plastic flow takes place in element t while maintaining the yield stress, equilibrium is possible only when strain reversal occurs in element c. This situation is just the reverse of that condition defined by Region 1. Referring to Fig. 5, the performance of the column in this region can be described by the relations

$$\Delta\delta' = -\frac{\alpha\lambda^2}{4}\Delta\epsilon_t \tag{3-1}$$

$$\Delta P = \Delta P_c \tag{3-2}$$

$$\Delta M_i = \Delta P_c b = \Delta P b \tag{3-3}$$

$$\Delta M_e = P_2\Delta\delta + \Delta P\delta_2 + \Delta P\Delta\delta. \tag{3-4}$$

Equating the increments of internal and external bending moments gives

$$P_2\Delta\delta' + \Delta P\delta_2' + \Delta P\Delta\delta' = \Delta P$$

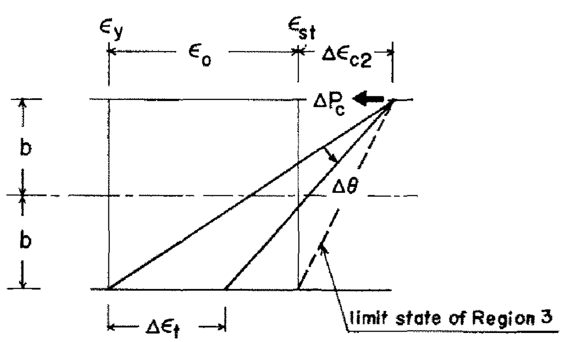


Fig. 5. Region 3.

or

$$\Delta P = \Delta P_c = \frac{P_2 \Delta \delta'}{1-\delta_2'-\Delta\delta'}. \tag{19}$$

Thus

$$P = P_2 + \Delta P = \left(\frac{1-\delta_2'}{1-\delta_2'-\Delta\delta'}\right) P_2 \tag{20}$$

and

$$\delta' = \delta_2' + \Delta\delta'. \tag{21}$$

Note that $\Delta\delta'$ has a negative sign in the above equations. Also note that if δ_i were to exceed b, the stress in the convex side element would become tension since b is the core radius of the section. This condition can never develop however in the present problem. Therefore, $\delta_i' = \delta_i/b$ must always be less than unity. Considering these two facts, it is clear that ΔP_c in Eq. (19) must be negative. Equations (19)–(21) are, of course, valid only until the strain in element t reaches the strain hardening point.

As seen in Fig. 5, the total increment of absolute deflection at the terminal point of Region 3 is equal to δ_1, i.e.

$$\Delta\delta_3' = -\delta_1'. \tag{22}$$

Substituting this value of $\Delta\delta_3'$ into Eq. (20) for $\Delta\delta'$, the terminal axial thrust for Region 3 is found to be

$$P_3 = \left(\frac{1-\delta_2'}{1-\delta_2'+\delta_1'}\right) P_2.$$

Substituting for P_2 from Eq. (17) and making use of Eq. (15), this equation reduces to the form

$$P_3 = \left(\frac{1}{1-\Delta\delta_2'}\right) P_y. \tag{23}$$

The terminal deflection in Region 3 is

$$\delta_3' = \delta_2' + \Delta\delta_3' = \delta_2' - \delta_1' = \Delta\delta_2'. \tag{24}$$

Making use of Eqs. (19) and (22), the decrease of axial thrust and thus the decrease of stress in element c is found to be

$$\Delta P_3 = \Delta P_{c\cdot 3} = -\frac{\delta_1'}{1-\delta_2'+\delta_1'} P_2 = -\frac{\delta_1'}{1-\Delta\delta_2'} P_2.$$

Substituting for P_2 from Eq. (17), one obtains

$$\Delta P_3 = -\frac{\delta_1' \Delta\delta_2'}{2\gamma\delta_2'(1-\Delta\delta_2')} P_y. \tag{25}$$

D. *Region 4—Compressive strain in element* t *increases into the strain hardening region while the compressive strain in element* c *increases again elastically*

The incremental changes which take place in Region 4 are shown in Fig. 6. One finds in this case that

$$\Delta\delta' = -\frac{\alpha\lambda^2}{4}\Delta\epsilon_t \tag{4-1}$$

$$\Delta P = \Delta P_c + \frac{AE_{st}}{2}\Delta\epsilon_t \tag{4-2}$$

$$\begin{aligned}\Delta M_i &= \left(\Delta P_c - \frac{AE_{st}}{2}\Delta\epsilon_t\right)b \\ &= (\Delta P - AE_{st}\Delta\epsilon_t)b \end{aligned} \tag{4-3}$$

$$\Delta M_e = P_3\Delta\delta + \Delta P\delta_3 + \Delta P\Delta\delta. \tag{4-4}$$

Equating the increments of internal and external bending moments gives

$$\Delta P - AE_{st}\Delta\epsilon_t = P_3\Delta\delta' + \Delta P\Delta\delta_3' + \Delta P\Delta\delta'.$$

Solving for ΔP, one obtains the relation

$$\Delta P = \frac{-(P_R - P_3)\Delta\delta'}{1-\delta_3'-\Delta\delta'}. \tag{26}$$

The axial thrust and deflection are found to be

$$P = P_3 + \Delta P = \frac{P_y - P_R\Delta\delta'}{1-\delta_3'-\Delta\delta'} \tag{27}$$

$$\delta' = \delta_3' + \Delta\delta'. \tag{28}$$

Note that $\Delta\delta'$ in this region is negative. The increment of stress in element c is given by

$$\Delta P_c = \Delta P - \frac{AE_{st}}{2}\Delta\epsilon_t = \Delta P + \frac{1}{2}P_R\Delta\delta' = -\frac{P_R}{2}\left[\frac{(1-2\gamma-\delta_3'^2)+(1-\delta_3')\Delta\delta'}{(1-\delta_3')(1-\delta_3'-\Delta\delta')}\right]\Delta\delta'. \tag{29}$$

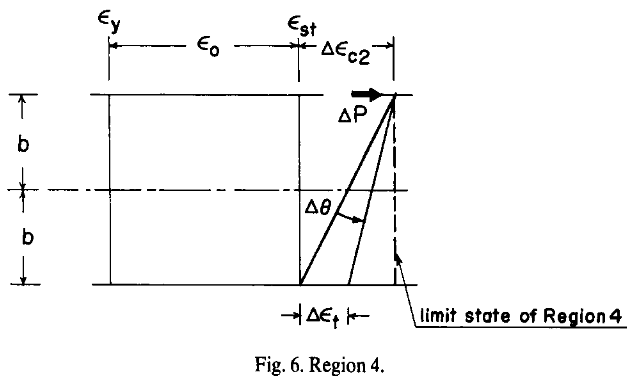


Fig. 6. Region 4.

If the term $(1-2\gamma-\delta_3'^2)$ is positive, ΔP_c as given by Eq. (29) will also be positive when $\Delta\delta'$ falls in the range

$$\frac{-(1-2\gamma-\delta_3'^2)}{(1-\delta_3')}<\Delta\delta'<0.$$

Using Eqs. (15) and (24), the term $(1-2\gamma-\delta_3'^2)$ can be written in the form

$$(1-2\gamma-\delta_3'^2)=(1-2\gamma)(1-\Delta\delta_2')+\delta_1'(\Delta\delta_2'+2\gamma). \tag{30}$$

It is clear that the right hand side of Eq. (30) must be positive. Therefore it is shown that the compressive stress in element c, which decreased elastically in Region 3, is now increasing elastically. Obviously, ΔP_c must have a peak value at some point in the above range for $\Delta\delta'$. This peak value and its corresponding value of $\Delta\delta'$ are found using the maximum condition $d(\Delta P_c)/d(\Delta\delta')=0$ which results in the relations

$$(\Delta\delta')_m=(1-\delta_3')-\sqrt{2(1-\delta_3'-\gamma)} \tag{31}$$

$$(\Delta P_c)_m=\frac{P_R}{2}\frac{[(1-\delta_3')-\sqrt{2(1-\delta_3'-\gamma)}]^2}{1-\delta_3'}. \tag{32}$$

If $(\Delta P_c)_m$ is larger than the decrease of stress which occurred from the limit state of Region 2 to the limit state of Region 3 as given by Eq. (25), the full strain reversal undergone in Region 3 will be recovered and the terminal point of Region 4 will be reached when the stress in both elements c and t arrives at the starting point of the virgin strain hardening region. This condition can be checked by calculating the value

$$D=(\Delta P_c)_m-\Delta P_3=\frac{P_R}{2}\left\{\frac{[(1-\delta_3')-\sqrt{2(1-\delta_3'-\gamma)}]^2}{1-\delta_3'}-\frac{\delta_1'\Delta\delta_2'}{\delta_2'(1-\Delta\delta_2')}\right\}.$$

Making use of Eqs. (15) and (24), this relation simplifies to

$$D=\frac{P_R}{2}\frac{[1-\sqrt{2(1-\delta_3'-\gamma)}]^2}{1-\delta_3'}. \tag{33}$$

From this form, it is obvious that D is positive; hence, it is shown that $(\Delta P_c)_m>\Delta P_3$. The condition that the strain reversal in element c which took place in Region 3 is just recovered, thus defining the terminal point of Region 4, is $\Delta P_3+\Delta P_c=0$ or

$$-\frac{P_R}{2}\left\{\frac{\delta_1'\Delta\delta_2'}{\delta_2'(1-\Delta\delta_2')}+\frac{[(1-2\gamma-\delta_3'^2)+(1-\delta_3')\Delta\delta']\Delta\delta'}{(1-\delta_3')(1-\delta_3'-\Delta\delta')}\right\}=0. \tag{34}$$

Making use of Eqs. (15) and (24), $\Delta\delta'$ to be used in Eq. (34) becomes equal to

$$\Delta\delta_4'=-\Delta\delta_2'=-\delta_3'. \tag{35}$$

Introducing this relation into Eqs. (27) and (28), one obtains

$$P_4 = P_y + P_R\delta_3' = P_y + P_R\Delta\delta_2' = P_y(1 + \Delta\delta_2'/\gamma) \tag{36}$$

$$\delta_4' = \delta_3' + \Delta\delta_4' = \delta_3' - \delta_3' = 0. \tag{37}$$

From Eq. (37), it is seen that at the terminal point of Region 4, the column has returned back to its original straight configuration.

E. *Region 5—Pure axial compression up to final failure*

Since the column is now straight, it will be subjected to pure compression in Region 5. The tangent modulus load in this state is given by substituting E_{st} for E in Eq. (2); thus,

$$P_T = \frac{2AE_{st}}{\alpha\lambda^2} = \frac{1}{2}P_R. \tag{38}$$

As soon as this critical load is reached, buckling will again take place accompanied by strain reversal in one flange element. The post buckling behavior can be analysed using procedures similar to those previously presented in which case one obtains

$$P = \frac{P_R}{2}\left(\frac{1+2\delta'}{1+\delta'}\right). \tag{39}$$

From this relation it is seen that as δ' approaches infinity, the column strength approaches P_R. Actually, the column strength can never exceed the value $P_u = A\sigma_u$ where σ_u is the maximum strength of the material as controlled by local buckling. If P_T is larger than P_u, the P vs δ' relation during the collapse state can be expressed in the form

$$P = \frac{P_u}{1+\delta'}. \tag{40}$$

In this case, the strain reversal which takes place in the convex side element is elastic.

EXAMPLES

The post buckling behaviors of several columns with different slenderness ratios, as predicted by the foregoing analysis, are illustrated in Fig. 7. In these examples, typical mild steel was assumed for the material and the mechanical properties were specified as follows [8]:

$$\epsilon_y = 1{\cdot}2 \times 10^{-3}, \quad \epsilon_0 = 10\epsilon_y, \quad \sigma_u = 1{\cdot}8\sigma_y, \quad E_{st} = 0{\cdot}03E.$$

For actual columns, the ratio of cell length to column length represents the ratio of the longitudinal dimension of the inelastic zone to the column length. This ratio was assumed rather arbitrarily equal to 0·4 in the examples.

From the results shown in Fig. 7, it is seen that columns with slenderness ratios smaller than some critical value can return back to their original straight position after having undergone limited amounts of buckling. The critical slenderness ratio is given by Eq. 14 which yields a numerical value of 11·4.

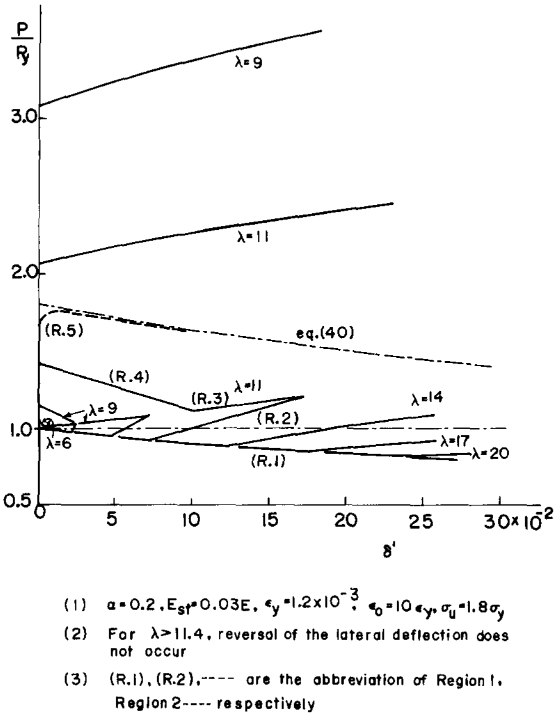


Fig. 7. Load–deflection curves.

In the present study, it was assumed that E_{st} remained constant throughout the post yield performance, even though, it is obvious that it decreases with increasing inelastic strain. Ratio $\alpha = a/L$ was also assumed to remain constant in the post yield range, even though, the inelastic zone expands considerably with increasing column load[9].

TEST RESULTS

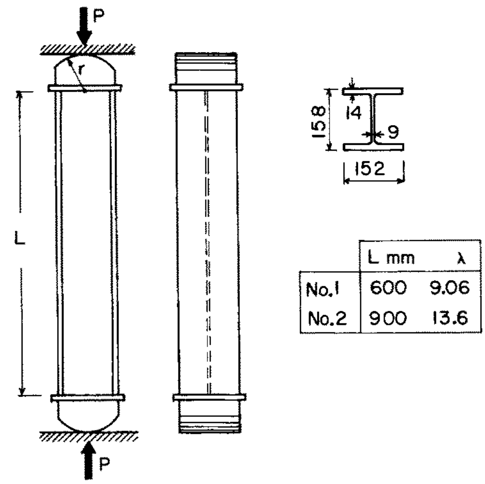
Tests were carried out on H-shaped steel columns as shown in Fig. 8. All test specimens were designed to buckle in the web direction. The average mechanical properties over the cross section were determined from stub-column compressive tests which gave the following results:

$$\sigma_y = 2{\cdot}67t/\text{cm}^2, \quad \sigma_u = 4{\cdot}2t/\text{cm}^2 = 1{\cdot}57\sigma_y, \quad \epsilon_y = 1270 \times 10^{-6},$$

$$\epsilon_{st} = 1270 \times 10^{-5} = 10\epsilon_y, \quad E_{st} = 74t/\text{cm}^2 = 0{\cdot}0352E.$$

These properties are similar to those adopted in the calculations of the foregoing examples.

To demonstrate the validity of basic theory presented herein, the test results of two column specimens having quite different behaviors are shown in Fig. 9. In these tests, no local buckling of plate elements or lateral torsional buckling was observed until the final collapse stage of the columns had been reached. Specimen No. 1 had a slight initial imperfection which may have caused the small lateral deflections which occurred in the early stages of loading. This imperfection may also have had some influence on the post buckling behavior.



	L mm	λ
No.1	600	9.06
No.2	900	13.6

Fig. 8. Test specimens.

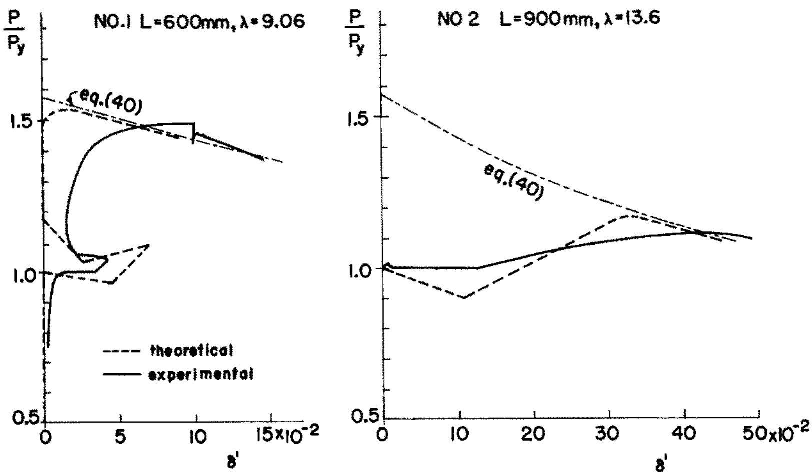


Fig. 9. Test results.

The theoretically predicted behaviors for these test specimens are shown in Fig. 9 by dashed lines. The ratio α was assumed equal to 0·2 in making these predictions. This value was selected because it gives an Euler load, Eq. 2 $P_E = 2AE/\alpha\lambda^2 = 10AE/\lambda^2$, which is nearly equal to $P_E = \pi^2 AE/\lambda^2 \fallingdotseq 10AE/\lambda^2$ which is the Euler load for the actual column.

Although the theoretical predictions are based on an extreme simplification of the problem, the general post buckling behaviors obtained show close resemblances to the actual column behaviors.

The major differences found between theoretical predictions and actual column behaviors are thought to come from the neglect of web participation, the uncertainty of estimating equivalent cell length, and the assumption of a constant strain hardening modulus, in making the theoretical predictions. It is believed that the participation of the web in actual columns makes the real post buckling behavior characteristics somewhat less pronounced than those predicted by theory. Obviously, the errors introduced by the assumption of a constant strain hardening modulus become large with increasing strain.

CONCLUSIONS

Although accurate quantitative predictions of actual column behavior cannot be made with the present theory which is based on a simplified model, the basic phenomena of column action in the post buckling region are qualitatively shown by the theory. This statement has been verified by the experimental evidence presented. Thus it may be concluded that

(1) The first critical load of columns having an Euler load much larger than the axial yield load is the axial yield load itself.

(2) Columns with slenderness ratios smaller than a certain critical value can return back to their original straight position after some initial buckling and their final failure will be governed by the smaller of the second tangent modulus load and the maximum strength load of the material.

(3) Columns with slenderness ratios larger than the critical value cannot experience a reversal of lateral deflection but some of the loss in axial load which takes place due to plastic flow in the early stage of buckling can be restored upon further buckling.

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